Rational Bargaining Solutions*

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Abstract

The von Neumann-Morgenstern axioms are uncontroversial desiderata for individual decision making. We say that a bargaining solution is rational, if it can be interpreted as the most preferred alternative under these axioms. Yet, we find that neither the Nash nor the Kalai-Smorodinsky bargaining solution is rational in this sense. We formalize two consequences of rationality, namely that one can neither be strictly better off nor strictly worse off from randomizing over different actions. These two axioms, together with other standard axioms, characterize the relative utilitarian bargaining solution. We then implement this bargaining solution in sub-game perfect equilibrium.

Keywords: axiomatic bargaining, relative utilitarian, rationality, implementation.

1 Introduction

The foundations of individual decision making in economics are laid by the axioms of von Neumann and Morgenstern (1944). While there is plenty evidence that these axioms do not always accurately describe the behavior of economic agents, there is little disagreement about their normative desirability. These axioms are widely accepted, and acting in accordance with them is often synonymous with rationality. We feel that

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rationality is equally desirable in the bargaining setting, where a bargaining solution prescribes what a group or arbitrator should choose. We define a bargaining solution as rational, if it can be interpreted as the most preferred alternative under some von Neumann-Morgenstern (vNM) preference relation. A consequence of the vNM axioms is that an agent would only randomize over different actions if she were indifferent between them. Consequently, there is no benefit of randomizing compared to choosing a deterministic action in the support. However, we find that the prominent bargaining solutions by Nash (1950) and Kalai and Smorodinsky (1975) violate this condition. Consider a scenario where two agents are bargaining over a single indivisible item and an arbitrator is appointed to choose an allocation. Both the Nash and Kalai-Smorodinsky (KS) bargaining solution would prescribe that the arbitrator flips a fair coin and allocates the item to the winner. However, neither solution selects the two deterministic allocations over which the coin randomizes. The prominent bargaining solutions demand that the arbitrator strictly prefers the coin-flip over the deterministic allocations. Clearly, this violates the vNM axioms.

In this paper, we aim to find a rational bargaining solution. To capture a central aspect of rationality, we propose the *no benefit of randomization* (NBR) axiom. This axiom relates non-convex bargaining sets to their convex hull. A non convex bargaining set represents a situation, where randomization is either not feasible or not permitted. Allowing for randomization means the group or arbitrator can choose from the convex hull of this set.[1] The axiom then says that if an alternative is selected by the bargaining solution in the non-convex set, it must still be selected in the convex hull of the set. Another consequence of the vNM axioms, and the flip-side of NBR, is that the agent is never strictly worse off when randomizing. Hence, when randomization is possible (i.e. the bargaining set is convex) and two alternatives are selected by the bargaining solution, then any mixture (i.e. convex combination) of these two alternatives must be selected as well. We call this axiom *convexity of the solution* (CS). We find that NBR and CS, together with standard axioms, characterize the *relative utilitarian* (RU) bargaining solution. The RU bargaining solution selects the alternatives with the highest sum of normalized utilities. Utilities are normalized such that the disagreement

[1] We assume here that individuals satisfy the von Neumann-Morgenstern axioms. This is a standard interpretation of the bargaining set, shared by both Nash (1950) and Kalai and Smorodinsky (1975).

point has utility 0 and the best alternative, among those that would be accepted by the others, has utility 1. The other axioms that underlie our characterization are invariance to affine transformations, strong Pareto, weak symmetry and weak IIA, which is a relaxation of Nash's independence of irrelevant alternatives axiom. The proof for two agents is simple and resembles the one of Nash (1950). The proof easily generalizes to any number of agents.

Reliance on randomization can be problematic for reasons other than the one outlined above. First, randomization might not be technologically feasible in a given situation. Second, even if feasible, it might not be transparent and might therefore be rejected by the agents. Third, randomization raises issues of commitment. An agent might agree to a lottery over the alternatives, but reject the outcome of the lottery if it isn't in her favor. Finally, randomization raises issues of ex-post fairness, because not only the lottery, but also the outcome of the lottery should satisfy normatively desirable postulates. These problems have sparked a large literature on the fair allocation of indivisible goods, which tries to find fair allocations without the use of randomization.[2] Note that if an arbitrator were to follow either the Nash or KS bargaining solution, he would have a desire to randomize in secret, when explicit randomization is prohibited. Our axiom NBR ensures that an arbitrator would have no desire to do so. This allows us to deal with problems of fair allocation, such as the division of an inheritance among descendants or the division of a household after a divorce, without relying on randomization.

Besides a characterization, we also implement the RU bargaining solution in subgame perfect equilibrium. This means that we identify a bargaining protocol, which in equilibrium leads to a RU-optimal alternative. We show that full implementation is not possible and identify a game form that weakly implements our bargaining solution. However, we are quite close to full implementation, as any RU-optimal alternative that is strictly better than the disagreement point for every agent is an equilibrium outcome. In this game, agents simultaneously make a proposal, consisting of an alternative and a claim how good this alternative is for each agent. In equilibrium, all agents propose

[2]For a recent survey of the literature see Amanatidis et al. (2022). Note that the typical axioms of this literature differ from the ones in the fair bargaining setting. Typical desiderata are no-envy, proportionality and their relaxations by Budish (2011).

the same RU-optimal alternative and this alternative is implemented immediately. If there is disagreement among the agents, the proposal with the highest reported sum of utilities has to be sequentially approved by all agents. Agents can choose whether to accept the alternative or receive a utility equal to the one that was reported. Hence, if the report was inflated, the alternative is rejected by at least one agent. If an agent rejects, all other agents receive 0 utility. Our game is similar to the ones by Moulin (1984) and Moore and Repullo (1988). The only other implementations of the RU bargaining solutions are by Miyagawa (2002) and Hagiwara (2020). They however only consider the case of two agents and strictly convex bargaining sets. Note that under strictly convex bargaining sets, there is a unique RU-optimal alternative. Our game on the other hand typically has multiple equilibrium outcomes, each corresponding to one of the multiple RU-optimal alternatives.

Other axiomatizations of the RU bargaining solution are by Pivato (2009) and Baris (2018). Pivato (2009) considers preferences over bargaining solutions and then imposes axioms on these preferences. This differs from the standard approach, established by Nash (1950), where axioms are imposed on the bargaining solution directly. Baris (2018) adapts the characterization of the utilitarian bargaining solution by Myerson (1981) to a utility-scale invariant setting. Their central axiom can be interpreted as a dynamic consistency condition. When facing uncertainty over what the bargaining set will be, the arbitrator makes a plan, which specifies for each possible bargaining problem a utility vector. Then the expected utility vector must be the solution in the expected bargaining set. Cao (1982) identifies necessary axioms for the RU bargaining solution but does not provide a characterization. Note that Cao (1982), Pivato (2009) and Baris (2018) assume the bargaining set to be convex, whereas we contribute to the literature on bargaining over non-convex sets (Kaneko, 1980; Zhou, 1997; Mariotti, 1998a,b; Conley and Wilkie, 1996; Denicolò and Mariotti, 2000; Ok and Zhou, 1999; Nagahisa and Tanaka, 2002; Xu and Yoshihara, 2006; Zambrano, 2016).

Related to the RU bargaining solution is a large literature on preference aggregation which characterizes a relative utilitarian rule (Karni, 1998; Dhillon and Mertens, 1999; Segal, 2000; Borgers and Choo, 2017; Marchant, 2019; Sprumont, 2019; Brandl, 2021; Peitler and Schlag, 2023). Especially related is Peitler and Schlag (2023). In an application of their aggregation rule, the RU bargaining solution is derived from the most preferred element of a menu-dependent social preference, where the menu consists of the alternatives which are better for every agent than the disagreement point.

Finally, there is a literature that considers the rationalizability of bargaining rules (Peters and Wakker, 1991; Bossert, 1994; Sánchez, 2000; Xu and Yoshihara, 2013). A bargaining rule is rationalizable if it can be interpreted as the most preferred alternative under a single preference relation over utility vectors, which applies independent of the bargaining problem. In this literature, rationality is understood as satisfying the weak axiom of revealed preference (or similar conditions). We take rationality to mean that the bargaining solution is consistent with the maximization of a von Neumann-Morgenstern preference, but we do not insist that it is the same preference for every bargaining problem.

We present the axioms and state the representation theorem in Section 2. The implementation is in Section 3. Section 4 concludes.

2 Axiomatization

Let $N := \{1, ..., n\}$ be a set of agents where $n \in \mathbb{N}$ and $n \geq 2$. A bargaining problem (S, d) consists of a bargaining set $S \subset \mathbb{R}^n$ and a disagreement point $d = (d_1, ..., d_n) \in S$. We denote by $\text{IR}(S, d) := \{(u_1, ..., u_n) \in S : u_i \geq d_i \text{ for all } i \in N\}$ the points in S that are *individually rational*, meaning weakly better than the disagreement point for every agent. We restrict attention to bargaining problems (S, d) where (i) $\text{IR}(S, d)$ is compact and (ii) for each $i \in N$ there exists a $u \in \text{IR}(S, d)$ such that $u_i > d_i$. We denote the domain of bargaining problems that satisfy these properties by $\mathcal{B}$. A bargaining solution f is a correspondence that assigns to every $(S, d) \in \mathcal{B}$ a non-empty subset of $\text{IR}(S, d)$.

Our central axiom is *no benefit of randomization*. For any $S \subset \mathbb{R}^n$, let $\text{conv}(S)$ denote the convex hull of S.

Axiom NBR (No Benefit of Randomization)**.** For every $(d, S) \in \mathcal{B}$,

$$f(S, d) \subseteq f(\text{conv}(S), d).$$

Note that NBR has no bite when the domain is restricted to convex bargaining sets, which is the classic setting of Nash (1950) and Kalai and Smorodinsky (1975). However, these popular bargaining solutions have been extended in various ways to domains that include non-convex problems. We find that these extensions violate NBR. In the following we demonstrate this for the extensions by Xu and Yoshihara (2006). Let f^{Nash} denote the Nash bargaining solution and f^{KS} denote the Kalai-Smorodinsky (KS) bargaining solution as in Xu and Yoshihara (2006). For $x \in \mathbb{R}$, let $\boldsymbol{x} := (x, ..., x) \in \mathbb{R}^n$.

Proposition 1. Both f^{Nash} and f^{KS} violate NBR.

Proof. Assume $n = 2$ and consider the barging problem $(S, \mathbf{0})$ where

$$S = \{(u_1, u_1) \in [0,1]^2 : u_1 \leq x \text{ or } u_2 \leq x\}$$

for some $x \in (0,1)$. Then $f^{\text{Nash}}(S, \mathbf{0}) = \{(1, x), (x, 1)\}$ and $f^{\text{KS}}(S, \mathbf{0}) = \{(x, x)\}$. However, $f^{\text{Nash}}(\text{conv}(S), \mathbf{0}) = f^{\text{KS}}(\text{conv}(S), \mathbf{0}) = \{(\frac{1+x}{2}, \frac{1+x}{2})\}$. Hence, NBR is violated. □

Another consequence of the von Neumann-Morgenstern axioms is that the arbitrator cannot be strictly worse off under randomization. If two utility vectors are both optimal from the eyes of the arbitrator, then a lottery over these vectors, assuming it is feasible, must also be optimal. This is captured by the following axiom.

Axiom CS (Convexity of the Solution)**.** For every $(S, d) \in \mathcal{B}$, if S is convex then so is $f(S, d)$.

Note that both the Nash and KS bargaining solution trivially satisfy this axiom, since these solutions are singletons whenever the bargaining set is convex.

Since the prominent bargaining solutions violate NBR and are therefore not rational, we are in need of an alternative solution. Besides the aforementioned axioms, this solution should satisfy agreed upon desiderata, which we outline next. The following two axioms are uncontroversial and underlie the solutions of both Nash (1950) and Kalai and Smorodinsky (1975).

Axiom PO (Pareto Optimality)**.** For every $(S, d) \in \mathcal{B}$, if $u \in f(S, d)$ then there is no $v \in S$ such that $v \neq u$ and $v_i \geq u_i$ for all $i \in N$.

We say that $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is a positive affine transformation if there exists $(\beta_i)_{i\in N} \in \mathbb{R}^n_+$ and $(\gamma_i)_{i\in N} \in \mathbb{R}^n$ such that for every $u \in \mathbb{R}^n$, $\alpha(u) = (\beta_1 u_1 + \gamma_1, ..., \beta_n u_n + \gamma_n)$. With slight abuse of notation we write $\alpha(S) := \{\alpha(u) \in \mathbb{R}^n : u \in S\}$ where $S \subset \mathbb{R}^n$.

Axiom INV (Invariance to affine transformations)**.** For every $(S, d) \in \mathcal{B}$ and positive affine transformation α,

$$f(\alpha(S), \alpha(d)) = \alpha(f(S, d)).$$

Another axiom that is shared by the Nash and KS bargaining solution is *symmetry*. We say that $S \subset \mathbb{R}^n$ is symmetric if for every $u \in S$, any permutation of u is in S as well. We say that a bargaining problem $(S, d) \in \mathcal{B}$ is symmetric if S is symmetric and if $d = \boldsymbol{x}$ for some $x \in \mathbb{R}$.

Axiom SYM (Symmetry)**.** For every $(S, d) \in \mathcal{B}$, if (S, d) is symmetric and $u \in f(d, S)$ then $u = \boldsymbol{x}$ for some $x \in \mathbb{R}$.

Note that SYM violates PO in non-convex environments. To see this, consider for instance $n = 2$ and the bargaining problem $(S, \mathbf{0}) \in \mathcal{B}$ where $S = \{\mathbf{0}, (1, 0.9), (0.9, 1)\}$. Then SYM would demand $f(S, \mathbf{0}) = \{\mathbf{0}\}$, clearly a violation of PO. Therefore, for domains that include non-convex bargaining sets, symmetry needs to be weakened.

Axiom WSYM (Weak Symmetry)**.** For every $(S, d) \in \mathcal{B}$, if (S, d) is symmetric then so is $f(S, d)$.

The fourth and final axiom differs in Nash (1950) and Kalai and Smorodinsky (1975). Nash (1950) assumes independence of irrelevant alternatives (IIA), while Kalai and Smorodinsky (1975) assume monotonicity. However, both the Nash and KS bargaining solution satisfy a weakening of IIA, which we call *weak IIA*. Furthermore, Xu and Yoshihara (2006) show that for domains including non-convex problems, montonicity can be replaced by weak IIA for the characterization of the KS bargaining solution. Hence, weak IIA is a common denominator of the popular solutions and so we will impose it as well. For any $(S, d) \in \mathcal{B}$ and $i \in N$ let $a_i(S, d) := \max\{u_i : u \in \mathrm{IR}(d, S)\}$. Furthermore, let $a(S, d) := (a_i(S, d))_{i\in N}$.

Axiom WIIA (Weak IIA)**.** For every $(S, d), (S', d') \in \mathcal{B}$, if $d' = d$, $S' \subset S$, $a(S', d') = a(S, d)$ and there exists $u \in f(S, d)$ such that $u \in S'$ then

$$f(S', d') = f(S, d) \cap S'.$$

We show that the above axioms characterize the *relative utilitarian* (RU) bargaining solution. Define f^{RU} such that for every $(S,d) \in \mathcal{B}$,

$$f^{\mathrm{RU}}(S,d) = \underset{u \in \mathrm{IR}(S,d)}{\arg\max} \sum_{i \in N} \frac{u_i - d_i}{a_i(S,d) - d_i}.$$

Theorem 1. f satisfies NBR, CS, PO, INV, WSYM and WIIA if and only if $f \equiv f^{\mathrm{RU}}$.

We sketch the proof for $n = 2$. A proof for a general number of agents is in Appendix A.

Proof. First, consider a bargaining problem $(S_x, \mathbf{0})$ where

$$S_x = \{\mathbf{0}, (1,0), (0,1), (1,x), (x,1)\}$$

for some $x \in [0,1]$. Note that $P_x := \{(1,x),(x,1)\}$ is the set of Pareto optimal points in S_x. Hence, by PO, $f(S_x, \mathbf{0}) \subseteq P_x$. Furthermore, note that $(S_x, \mathbf{0})$ is symmetric. So by WSYM,

$$f(S_x, \mathbf{0}) = P_x = \{(1,x),(x,1)\}. \tag{1}$$

Second, consider the bargaining problem $(\mathrm{conv}(S_x), \mathbf{0})$. By (1) and NBR, $P_x \subseteq f(\mathrm{conv}(S_x), \mathbf{0})$. As $(\mathrm{conv}(S_x), \mathbf{0})$ is convex, $\mathrm{conv}(P_x) \subseteq f(\mathrm{conv}(S_x), \mathbf{0})$ by CS. Note that $\mathrm{conv}(P_x) = \{u \in [0,1]^2 : u_1 + u_2 = x\}$ and $\mathrm{conv}(S_x) = \{u \in [0,1]^2 : u_1 + u_2 \le x\}$. Hence, $\mathrm{conv}(P_x)$ is the set of Pareto-optimal points in $\mathrm{conv}(S_x)$ and by PO,

$$f(\mathrm{conv}(S_x), \mathbf{0}) = \left\{u \in [0,1]^2 : u_1 + u_2 = x\right\}. \tag{2}$$

Third, consider any *normalized* bargaining problem $(S,d) \in \mathcal{B}$, meaning a bargaining problem where $d = \mathbf{0}$, $\mathrm{IR}(S,\mathbf{0}) = S$ and $a(S,\mathbf{0}) = \mathbf{1}$. Let $x^* := \max_{u \in S}(u_1 + u_2)$, which must exist due to our assumption of compactness. Note that $S \subseteq \mathrm{conv}(S_{x^*})$ and that $S \cap f(\mathrm{conv}(S_{x^*}), \mathbf{0})$ is non-empty. Hence, (2) and WIIA imply

$$f(S, \mathbf{0}) = S \cap f(\mathrm{conv}(S_{x^*}), \mathbf{0}) = \underset{u \in S}{\arg\max}(u_1 + u_2). \tag{3}$$

Fourth, note that for any $(S,d) \in \mathcal{B}$ with $\mathrm{IR}(S,\mathbf{0}) = S$ there exists a normalized bargaining problem that can be reached through a positive affine transformation. Specifically, for

$$\alpha(u) = \left(\frac{u_i - d_i}{a_i(S,d) - d_i}\right)_{i \in N},$$

$(\alpha(S), \alpha(d)) \in \mathcal{B}$ is a normalized bargaining problem. Then by (3) and INV,

$$f(S,d) = \underset{u \in \mathrm{IR}(S,d)}{\arg\max} \left(\frac{u_1 - d_1}{a_1(S,d) - d_1} + \frac{u_2 - d_2}{a_2(S,d) - d_2} \right). \tag{4}$$

Finally, consider any $(S,d) \in \mathcal{B}$. A consequence of WIIA is $f(S,d) = f(\mathrm{IR}(S,d), d)$. This is because $f(S,d) \subseteq \mathrm{IR}(S,d)$ by definition and hence $f(\mathrm{IR}(S,d), d) = f(S,d) \cap \mathrm{IR}(S,d) = f(S,d)$. Therefore, (9) applies to any $(S,d) \in \mathcal{B}$.

□

3 Implementation

Let A be a set of alternatives and let Θ be a set of states. For every $\theta \in \Theta$, let $u_i^\theta : A \to \mathbb{R}$ denotes Agent's i normalized vNM utility function over A. We designate one alternative in A as the disagreement alternative and denote it by a_{dis}. For every $\theta \in \Theta$, (S_θ, d_θ) is the bargaining problem associated with θ where $S_\theta := \{(u_1^\theta(a), ..., u_n^\theta(a)) \in \mathbb{R}^n : a \in A\}$ and $d_\theta := (u_1^\theta(a_{\mathrm{dis}}), ..., u_n^\theta(a_{\mathrm{dis}}))$. For any $\theta \in \Theta$, we say that $a \in A$ is *RU-optimal* under θ if $(u_1^\theta(a), ..., u_n^\theta(a)) \in f^{\mathrm{RU}}(S_\theta, d_\theta)$.

We consider a social planner who wants to bring about an RU-optimal alternative for the group. We assume that the planner doesn't know the state, but the agents do. Since the planner doesn't know the state, he doesn't know which alternative is RU-optimal and hence he cannot simply impose an alternative on the group. Instead, the planner has to specify a game form, which, for every state, leads to an RU-optimal alternative in equilibrium. For a game form g and any $\theta \in \Theta$, we denote by g_θ the game with the game form g and agents' preferences according to u_1^θ to u_n^θ. We say that a game form g *fully implements* the RU bargaining solution in sub-game perfect equilibrium (SPE) if for every $\theta \in \Theta$ and $a \in A$, a is RU-optimal under θ if and only if a is an SPE outcome of g_θ. Unfortunately, full implementation is not possible for two players and a domain as specified in Section 2.

Proposition 2. Let $n = 2$ and assume that for every $(S,d) \in \mathcal{B}$ there exists a $\theta \in \Theta$ such that $(S,d) = (S_\theta, d_\theta)$. Then there doesn't exist an extensive game form g that fully implements the RU bargaining solution.

A formal proof is in Appendix B. The intuition for this result goes as follows. Consider a normalized bargaining problem $(S, \mathbf{0})$ for two agents where both $(1,0)$ and $(0,1)$ are in $f^{\mathrm{RU}}(S, \mathbf{0})$. An example of such a bargaining problem is the division of a dollar among risk neutral agents. In that case any efficient division is RU-optimal, including allocating the entire dollar to one of the agents. Now consider a state θ corresponding to such a bargaining problem and a game form g that implements the solution. Then there must exist two strategy profiles s^+, s^- that are SPE of g_θ and where $u^\theta(s^+) := (u_1^\theta(s^+), u_2^\theta(s^+)) = (1, 0)$ and $u^\theta(s^-) = (0, 1)$. Since by assumption 0 is the minimal utility for both players, $u_1^\theta(s_1, s_2^-) = 0$ for all s_1 and $u_2^\theta(s_1^+, s_2) = 0$ for all s_2. This in turn implies that (s_1^+, s_2^-) is a Nash equilibrium with pay-offs $(0, 0)$. In the formal proof we then use (s_1^+, s_2^-) to construct an SPE with pay-offs $(0, 0)$. Since, $(0, 0)$ is not in $f^{\mathrm{RU}}(S_\theta, d_\theta)$, the RU bargaining solution cannot be fully implemented.

One possible way to proceed would be to exclude all states that correspond to bargaining problems as the ones described above. However, we feel that bargaining problems such as the division of a dollar among among risk neutral agents are canonical problems that shouldn't be ignored. Similarly, bargaining among two agents is a canonical setting that shouldn't be ignored either. Hence, we decided on the following remedy. We assume that at least one of the RU-optimal alternatives is strictly better than a_{dis} for every agent. We call such alternatives *strictly RU-optimal*. Then we weakly implement the RU bargaining solution in SPE. We say that a game form g *weakly implements* the RU bargaining solution in SPE if for every $\theta \in \Theta$, every SPE outcome of g_θ is RU-optimal. In fact, for the game form we identify, every strictly RU-optimal alternative will be an SPE outcome. Hence, we are quite close to full implementation. The restrictions we impose on Θ are as follows.

Assumption A1. For every $\theta \in \Theta$, $(S_\theta, d_\theta) \in \mathcal{B}$ and (S_θ, d_θ) is a normalized bargaining problem.

Assumption A2. For every $\theta \in \Theta$, there exists an $a \in A$ such that a is strictly RU-optimal under θ.

Assumption A3. For every $\theta \in \Theta$ and $i \in N$, there exists a $b_i \in A$ such that $u_i^\theta(b_i) = 1$ and $u_j^\theta(b_i) = 0$ for all $j \in N \setminus \{i\}$.

We now define the game form that weakly implements the RU bargaining solution under A1 to A3. We denote the game form by g^*. The game form has two stages.

Stage 1: Each agent $i \in N$ simultaneously makes a proposal $p^i = (a^i, v^i) \in A \times [0,1]^n$, consisting of an alternative a^i and a report about the alternative's normalized utility for each agent $v^i = (v_1^i, ..., v_n^i)$. There are three possibilities to consider:

(1.1) All agree, i.e. $p^i = (a, v)$ for all $i \in N$. Then a is implemented.

(1.2) All but one agree, i.e. $p^i = (a, v)$ for all $i \in N \setminus \{j\}$ and $p^j \neq (a, v)$. We call j the *dissenting* agent. If $\sum_{i \in N} v_i = \sum_{i \in N} v_i^j$ then a_{dis} is implemented. Otherwise, go to Stage 2.

(1.3) Neither (1.1) nor (1.2) applies. Then the agent with the lowest, but strictly positive sum of reported utilities can choose an alternative.

Stage 2: Let P_+ denote the set of proposals that report strictly positive utility for all individuals and let P_0 denote the set of proposals that do not. Let $p^* = (a^*, v^*)$ denote the proposal in P_+ with the highest sum of reported utilities. If P_+ is empty, then let (a^*, v^*) denote the proposal in P_0 with the highest sum of reported utilities. Each agent sequentially chooses "accept" or "reject". If Agent i rejects, then with probability v_i^* Agent i can choose an alternative and with probability $1 - v_i^*$ the alternative a_{dis} is implemented. If all accept, then a^* is implemented. The order is determined as follows. Let j denote the dissenting agent in Stage 1. If $p^j = p^*$, then Agent j decides last and if $p^j \neq p^*$, then Agent j decides first.[3] Otherwise, agents are ordered by their index.

Theorem 2. If A1, A2 and A3 are satisfied, then g^* weakly implements the RU bargaining solution in SPE. Furthermore, every strictly RU-optimal alternative is an SPE outcome.

We prove the second part of the theorem below. The proof of the first part is in Appendix C.

Proof. Fix some $\theta \in \Theta$. In the following we omit θ on the individual utility functions. We show that every strictly RU-optimal alternative is an SPE outcome. Assume $a \in A$ is strictly RU-optimal. In the first stage, all agents report $(a, u(a))$. This leads

[3]Note that this description is consistent with $n = 2$, in which case the agent with the lower sum of reported utilities chooses first.

to alternative a. Now assume some Agent j deviates to (a^j, v^j). Agent j would not choose v^j such that $\sum_{i\in N} u_i(a) = \sum_{i\in N} v_i^j$, as this would lead to a_{dis}. So consider $\sum_{i\in N} u_i(a) > \sum_{i\in N} v_i^j$. Then $p^* = (a, u(a))$ and Stage 2 has an SPE where all accept. Hence, the outcome is a, which is no improvement for Agent j. Alternatively, consider $\sum_{i\in N} u_i(a) < \sum_{i\in N} v_i^j$. Then $p^* = (a^j, v^j)$ and Stage 2 has an SPE where each $i \in N$ chooses b_i after rejecting and where the first agent rejects. This is an SPE because at least for one Agent i it holds that $u_i(a^j) < v_i^j$ and hence this agent is strictly better off rejecting. The first agent to choose anticipates this and rejects immediately. Since Agent j is not the first to choose, her utility is 0. By the above argument, no agent has an incentive to deviate from $(a, u(a))$ in Stage 1. $\square$

4 Conclusion

This paper considers the implications of the von Neumann-Morgenstern axioms on fair bargaining. We find that these axioms are inconsistent with the prominent bargaining solutions by Nash (1950) and Kalai and Smorodinsky (1975) and offer an alternative, the relative utilitarian bargaining solution. On first sight, this solution might not seem very fair. For instance, when dividing a dollar among risk neutral agents, giving the entire dollar to one of the agents is permitted by the solution. However, fairness can be understood as treating symmetric agents in the same way, giving them them equal consideration. This is captured by the weak symmetry axiom. If giving the entire dollar to Agent 1 is permitted, then it must also be permitted to give the entire dollar to Agent 2. However, we do not insist on equality. This allows us to make trade-offs. The relative utilitarian bargaining solution is willing to improve one agent's utility if it comes at a smaller cost to another agent.

Appendix A

We begin with a definition of S_x and P_x for a general $n \in \mathbb{N}$. For this purpose, the following notation is introduced. For $x \in [1, n]$, let $\lfloor x \rfloor := \max\{k \in \{1, ..., n\} : k \leq x\}$, meaning $\lfloor x \rfloor$ is x rounded down to the closest integer. For any $u \in \mathbb{R}^n$, let $\pi(u) \subset \mathbb{R}^n$ denote the set containing u and all its permutations. For any $k \in \{0, ..., n\}$, let $g(k)$

denote the sequence of length n where $g_i(k) = 1$ if $i \le k$ and $g_i(k) = 0$ otherwise. Hence, $g(0) = \mathbf{0}$, $g(1) = (1, 0, ..., 0)$ and so on. For any $x \in [1, n]$, let $h(x)$ denote the sequence of length n where $h_i(x) = 1$ if $i \le x$, $h_i(x) = x - \lfloor x \rfloor$ if $i = \lfloor x \rfloor + 1$ and $h_i(k) = 0$ otherwise. For any $x \in [1, n]$, let

$$P_x := \pi(h(x)), \qquad\qquad S_x := P_x \cup \bigcup_{k=0}^{\lfloor x \rfloor} \pi(g(k)).$$

The following lemma identifies $\text{conv}(S_x)$ and $\text{conv}(P_x)$.

Lemma 1. For any $x \in [1, n]$, $\text{conv}(S_x) = \{u \in [0, 1]^n : \sum_{i \in N} u_i \le x\} =: R_x$ and $\text{conv}(P_x) = \{u \in [0, 1]^n : \sum_{i \in N} u_i = x\} =: Q_x$.

Proof. Note that $S_x \subseteq R_x$ and $P_x \subseteq Q_x$. We prove the lemma by showing that S_x (resp. P_x) contains all extreme points of R_x (resp. Q_x). Then the lemma follows from the Krein–Milman theorem. An element $u \in S \subset \mathbb{R}^n$ is an *extreme point* of S if there doesn't exist $v, w \in S$ and $\lambda \in (0, 1)$ such that $v \neq w$ and $u = \lambda v + (1 - \lambda)w$.

First, we show that an extreme point u of R_x (resp. Q_x) can have at most one coordinate u_j such that $0 < u_j < 1$. Consider $u \in R_x$ (resp. Q_x) with $0 < u_j < 1$ and $0 < u_k < 1$ for some $j \neq k$. Then there exists $v, w \in R_x$ (resp. Q_x) and $\varepsilon \in (0, 1]$ such that $v_i = w_i = u_i$ whenever $i \notin \{j, k\}$ and

$$\begin{aligned} v_j &= u_j + \varepsilon, & w_j &= u_j - \varepsilon, \\ v_k &= u_k - \varepsilon, & w_k &= u_k + \varepsilon. \end{aligned} \tag{5}$$

Since $\frac{1}{2}v + \frac{1}{2}w = u$, u is not an extreme point of R_x (resp. Q_x).

Second, we show that if there is an extreme point of R_x with exactly one coordinate u_j such that $0 < u_j < 1$ then $u \in \pi(h(x))$. Assume $u \notin \pi(h(x))$ and $0 < u_j < 1$. Then there exists $v, w \in R_x$ and $\varepsilon \in (0, 1]$ such that $v_i = w_i = u_i$ whenever $i \neq j$ and

$$v_j = u_j + \varepsilon, \qquad\qquad w_j = u_j - \varepsilon.$$

Since $\frac{1}{2}v + \frac{1}{2}w = u$, u is not an extreme point of R_x.

By the above arguments, the only candidates for extreme points of R_x (resp. Q_x) are the points in S_x (resp. P_x).

$\square$

We now present the proof of Theorem 1. The proof is nearly identical to the sketch in Section 2. Nevertheless, we state the proof for sake of completeness.

Proof of Theorem 1

First, consider a bargaining problem $(S_x, \mathbf{0})$ for some $x \in [1, n]$. Note that P_x is the set of Pareto optimal points in S_x. Hence, by PO, $f(S_x, \mathbf{0}) \subseteq P_x$. Furthermore, note that $(S_x, \mathbf{0})$ is symmetric. So by WSYM,

$$f(S_x, \mathbf{0}) = P_x. \tag{6}$$

Second, consider the bargaining problem $(\text{conv}(S_x), \mathbf{0})$. By (6) and NBR, $P_x \subseteq f(\text{conv}(S_x), \mathbf{0})$. As $(\text{conv}(S_x), \mathbf{0})$ is convex, $\text{conv}(P_x) \subseteq f(\text{conv}(S_x), \mathbf{0})$ by CS. From Lemma 1 we can see that $\text{conv}(P_x)$ is the set of Pareto-optimal points in $\text{conv}(S_x)$. Then by PO,

$$f(\text{conv}(S_x), \mathbf{0}) = \left\{ u \in [0,1]^n : \sum_{i \in N} u_i = x \right\}. \tag{7}$$

Third, consider any *normalized* bargaining problem $(S, d) \in \mathcal{B}$, meaning a bargaining problem where $d = \mathbf{0}$, $\text{IR}(S, \mathbf{0}) = S$ and $a(S, \mathbf{0}) = \mathbf{1}$. Let $x^* := \max_{u \in S} \sum_{i \in N} u_i$, which must exist due to our assumption of compactness. Note that $S \subseteq \text{conv}(S_{x^*})$ and that $S \cap f(\text{conv}(S_{x^*}), \mathbf{0})$ is non-empty. Hence, (7) and WIIA imply

$$f(S, \mathbf{0}) = S \cap f(\text{conv}(S_{x^*}), \mathbf{0}) = \arg\max_{u \in S} \sum_{i \in N} u_i. \tag{8}$$

Fourth, note that for any $(S, d) \in \mathcal{B}$ with $\text{IR}(S, \mathbf{0}) = S$ there exists a normalized bargaining problem that can be reached through a positive affine transformation. Specifically, for

$$\alpha(u) = \left(\frac{u_i - d_i}{a_i(S, d) - d_i} \right)_{i \in N},$$

$(\alpha(S), \alpha(d)) \in \mathcal{B}$ is a normalized bargaining problem. Then by (8) and INV,

$$f(S, d) = \arg\max_{u \in \text{IR}(S,d)} \left(\sum_{i \in N} u_i \frac{u_i - d_i}{a_i(S, d) - d_i} \right). \tag{9}$$

Finally, consider $(S, d) \in \mathcal{B}$, possibly with $\text{IR}(S, \mathbf{0}) \neq S$. A consequence of WIIA is $f(S, d) = f(\text{IR}(S, d), d)$. This is because $f(S, d) \subseteq \text{IR}(S, d)$ by definition and hence $f(\text{IR}(S, d), d) = f(S, d) \cap \text{IR}(S, d) = f(S, d)$. Therefore, (9) applies to any $(S, d) \in \mathcal{B}$. This concludes the proof.

Appendix B

We follow the notation of Moore and Repullo (1988). Consider a two player extensive game form g. Let T denote the set of nodes. For any $t \in T$ we denote by $g(t)$ the sub-game starting at node t. For any $t \in T$ and $i \in \{1, 2\}$ we denote by $\mathcal{S}_i(t)$ the set of actions of Player i. We assume that $|\mathcal{S}_i(t)| \geq 1$ for all $t \in T$. If $|\mathcal{S}_i(t)| = 1$, then Player i has no decision at t. If both $|\mathcal{S}_1(t)| > 1$ and $|\mathcal{S}_2(t)| > 1$ then both players move simultaneously at t. Let $\mathcal{S}_i := \bigtimes_{t\in T} \mathcal{S}_i(t)$ denote the strategy space of Player i and let $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ denote the set of strategy profiles. For any $s \in \mathcal{S}$ and $t \in T$ we denote by $s|t$ the part of s that specifies the strategy profile for the game $g(t)$. For any $s \in \mathcal{S}$ and $t \in T$ we denote by $s(t) \in \mathcal{S}_1(t) \times \mathcal{S}_2(t)$ the action pair that s prescribes for the node t. Terminal nodes are alternatives in A. For any $s \in \mathcal{S}$ and $\theta \in \Theta$, $u^\theta(s) = (u_1^\theta(s), u_2^\theta(s))$ denotes the utility vector of the terminal node that results from s. For any $s \in \mathcal{S}$ and $t \in T$ we write $u^\theta(s|t)$ to denote the utility vector of the terminal node that is reached when starting at t and playing according to s.

Proof of Proposition 2

Fix some $\theta \in \Theta$ such that $(S(\theta), d(\theta))$ is a normalized bargaining problem and both $(1, 0)$ and $(0, 1)$ are in $f^{\text{RU}}(S(\theta), d(\theta))$. In the following we omit θ on the individual utility functions and the game. We have already established in the main section that there must be $s^+, s^- \in \mathcal{S}$ such that both are SPE of g with $u(s^+) = (1, 0)$ and $u(s^-) = (0, 1)$ and that $(s_1^+, s_2^-) =: s^0$ is a NE with $u(s^0) = (0, 0)$. In the following we construct $s^* \in \mathcal{S}$ such that s^* is an SPE with $u(s^*) = (0, 0)$.

Let $t_0, t_0, ..., t_k, t_{k+1}$ denote the nodes of the equilibrium path of s^0, where t_0 is the initial node of g and t_{k+1} is a terminal node of g associated with the pay-off $(0, 0)$. Consider the second to last node t_k. Let $(x_k, y_k) \in \mathcal{S}_1(t_k) \times \mathcal{S}_2(t_k)$ denote the action pair that leads to t_{k+1}, formally $t_{k+1} = (t_k, (x_k, y_k))$. If there are only terminal nodes succeeding t_k, then $g(t_k)$ is a one-stage game and $s^0|t_k$ is not only a NE of $g(t_k)$ but also an SPE. Hence, $s^*|t_k = s^0|t_k$ ensures that $s^*|t_k$ is an SPE of $g(t_k)$ with outcome $(0, 0)$. If there are non-terminal nodes succeeding t_k, then we construct $s^*|t_k$ as follows.

Choose $s^*(t_k) = (x_k, y_k)$. For any (x_k, y) with $y \in \mathcal{S}_2(t_k)$ choose $s^*|(t_k, (x_k, y)) = s^+|(t_k, (x_k, y))$. This ensures that $s^*|(t_k, (x_k, y))$ is an SPE of $g(t_k, (x_k, y))$ for all

$y \in \mathcal{S}_2(t_k)$. Furthermore, it must be that $u_2(s^*|(t_k,(x_k,y))) = 0$ for all $y \in \mathcal{S}_2(t_k)$, because $(t_k,(x_k,y))$ can be reached by a unilateral deviation of Player 2 in the strategy profile s^+. Similarly, choose $s^*|(t_k,(x,y_k)) = s^-|(t_k,(x,y_k))$ for all $x \in \mathcal{S}_1(t_k)$. For $s^*|(t_k,(x,y))$ such that neither $x = x_k$ nor $y = y_k$ choose an arbitrary SPE. Note that $s^*|t_k$ has been constructed such that an SPE is played at all nodes succeeding t_k, such that the outcome is $(0,0)$ and such that no Player has an incentive to deviate at t_k. Therefore, $s^*|t_k$ is an SPE of $g(t_k)$ with outcome $(0,0)$.

Next consider t_l for any $l \in \{0,...,k-1\}$. Let $(x_l,y_l) \in \mathcal{S}_1(t_l) \times \mathcal{S}_2(t_l)$ denote the decision that leads to t_{l+1}, formally $t_{l+1} = (t_l,(x_l,y_l))$. Assume $s^*|t_{l+1}$ is an SPE of $g(t_{l+1})$ with outcome $(0,0)$. Choose $s^*(t_l) = (x_l,y_l)$ and construct $s^*|(t_l,(x,y))$ for $(x,y) \neq (x_l,y_l)$ just as before. Then $s^*|t_l$ is an SPE of $g(t_l)$ with outcome $(0,0)$. By induction, s^* is an SPE of g with outcome $(0,0)$. This concludes the proof.

Appendix C

Proof of Theorem 2, First Part

We prove that for any $\theta \in \Theta$, every SPE outcome is RU-optimal under θ. Fix some $\theta \in \Theta$. In the following we omit θ on the individual utility functions. We assume a is an SPE outcome which is not RU-optimal and derive a contradiction.

First, consider the case (1.3) where a is chosen by an agent. Then at least one Agent i prefers b_i to a. This agent can deviate to a proposal with the lowest, but strictly positive sum of reported utilities among the other proposals and then choose b_i.

Second, consider the case (1.1) where a is implemented through some unanimous proposal (a,v). Note that at least one Agent j prefers some strictly RU-optimal alternative a'. If $v_i \leq u_i(a)$ for all $i \in N$ then this agent can deviate to $p^j = (a',(u_1(a')-\varepsilon,...,u_n(a')-\varepsilon))$ for ε sufficiently small such that $p^* = p^j$. Then the unique SPE outcome of Stage 2 is that all agents accept and a' is implemented, hence j has a profitable deviation. Alternatively, if $v_j > u_j(a)$ for some $j \in N$, then j can deviate to a proposal (a^j,v^j) with $\sum_{i\in N} v_i^j < \sum_{i\in N} v_i$ and be the first to reject in Stage 2, which gives utility v_j.

Finally, consider the case (1.2). There are three sub-cases to consider. First,

$a = a_{\text{dis}}$ because $\sum_{i \in N} v_i = \sum_{i \in N} v_i^j$. If $n \geq 3$, then some Agent i can deviate in Stage 1 and choose b_i. So assume $n = 2$. If $v_1^1 + v_2^1 = v_1^2 + v_2^2 > 1$, then an agent would be better off to propose a lower sum, be the first to choose in Stage 2 and then reject, yielding strictly positive utility. If $v_1^1 + v_2^1 = v_1^2 + v_2^2 \leq 1$, then for some strictly RU-optimal alternative a' an agent can propose $(a', (u_1(a') - \varepsilon, u_2(a') - \varepsilon))$ as earlier, leading to a'.

Second, a is implemented through acceptance by all agents. This implies that $\sum_{i \in N} v_i^* \leq \sum_{i \in N} u_i(a)$, as otherwise one agent would be better off to reject. Furthermore, there must be an Agent k that strictly prefers an RU-optimal alternative a'. If $k = j$, then j can deviate to $p^j = (a', (u_1(a') - \varepsilon, ..., u_n(a') - \varepsilon))$ as earlier, leading to a'. If $k \neq j$, then k can deviate to be to be the one with the lowest, but strictly positive sum of reported utilities and then choose a'.

Third, a is chosen after some agent rejects. Let $\sigma := \sum_{i \in N} u_i(a')$ for any RU-optimal alternative a'. Assume $n = 2$ and let $p^1 = (a^1, (v_1, v_2))$ and $p^2 = (a^2, (w_1, w_2))$. Consider the case where $p^* = p^1$, $w_1 + w_2 \geq \sigma$ and $w_1, w_2 > 0$. Then Agent 2 decides first in Stage 2, rejects and with probability v_2 chooses a with $u(a) = (x, 1)$ for some $x \in [0, 1)$. Agent 1 has an incentive to deviate to a proposal with a lower sum, unless

$$w_1 \leq v_2 x. \tag{10}$$

Furthermore, since $w_1 + w_2 \geq \sigma$,

$$w_1 + w_2 \geq x + 1. \tag{11}$$

Then (10) and (11) implies $w_2 = 1$ and either $x = 0$ or $v_2 = 1$. But $x = 0$ is not possible since $w_1 > 0$. Hence $w_1 = x$ and furthermore $1 + x = \sigma$. We have therefore shown that a is RU-optimal and that it is implemented with certainty, contradicting the assumption that a is not RU-optimal. Next consider the case where $p^* = p^1$ and either $w_1 + w_2 < \sigma$ or $\min\{w_1, w_2\} = 0$. Again Agent 2 decides first in Stage 2, rejects and with probability v_2 chooses a with $u(a) = (x, 1)$ for some $x \in [0, 1)$. Note that Agent 1 can implement any strictly RU-optimal alternative a' by deviating to the proposal $(a', (u_1(a') - \varepsilon, u_2(a') - \varepsilon))$. The only case in which Agent 1 would have no incentive to do so, is if $v_2 = 1$ and if a is the best strictly RU-optimal alternative for Agent 1, which contradicts the initial assumption. Note that $p^* = p^1$ is without loss of generality, because the analogous argument can be made for $p^* = p^2$. Therefore the case $n = 2$ is completely covered. The argument extends to a general n for when $p^* = p^j$ where j is the dissenting agent. If $p^* = p^i$ for $i \neq j$, then i can deviate to induce case (1.3) and choose b_i. This concludes the proof of the first part of Theorem 2.

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